INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 41, Northern Autumn 2019 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An illusionist lays a full deck of 52 cards in a row and tells spectators that 51 cards will be taken away step by step with only the Three of Clubs remaining on the table. On each step some spectator tells the illusionist a number so that a card lying on the place with this number in the row is taken away. However, the illusionist makes his own decision from which side of the row, left or right, he should count that number from to take the card away. For which initial positions of the Three of Clubs can the illusionist guarantee the success of his trick for sure? (3 points)

2. Let $ABCDE$ be a convex pentagon such that AE is parallel to CD and $AB = BC$. Let the angle bisectors of angles A and C intersect at point K. Prove that BK is parallel to AE. (4 points)

3. An integer x written on a blackboard can be replaced either with $3x + 1$ or with $\lfloor x/2 \rfloor$ (the greatest integer not exceeding $x/2$). Prove that if 1 is initially written, then any positive integer can be obtained by using the operations above. (4 points)

4. In a polygon, any two adjacent sides are perpendicular. Two vertices of the polygon are called *unfriendly*, if the angle bisectors emanating from those vertices are perpendicular. Prove that for each vertex the number of vertices unfriendly with that vertex is even. (5 points)

5. There is a row of 100 squares each containing a counter. Any 2 neighbouring counters can be swapped for 1 dollar and any 2 counters that have exactly 4 counters between them can be swapped for free. What is the least amount of money that must be spent to rearrange the counters in reverse order? (5 points)

O Level Senior Paper Solutions

Prepared by Oleksiy Yevdokimov and Greg Gamble

1. The illusionist can guarantee the success of his trick when the Three of Clubs is placed at either end of the row of 52 cards. Indeed, the illusionist can be forced to take away the Three of Clubs only if it is a central (symmetrical with respect to both ends of the row) card at some step. Otherwise he always has the opportunity to remove another card. If the Three of Clubs is placed at either end of the row of 52 cards, it can only become a central card on the very last turn, allowing the illusionist to guarantee the success of his trick.

   We claim that for any other position the spectators can prevent the success of the trick. Suppose that the Three of Clubs is placed at an interior position of the row of 52 cards, i.e. not at either end. Here are two possible strategies for spectators.

   Strategy 1. The spectators keep nominating numbers that match interior positions. This guarantees that the illusionist cannot remove a card from either end. When there only 3 cards remaining in the row, the Three of Clubs is necessarily the middle card. The spectators should then nominate number 2 and force the illusionist to remove the Three of Clubs.

   Strategy 2. The spectators always nominate the number of the position of the Three of Clubs. Then the illusionist is forced to remove the card in the mirror symmetric position, decreasing the larger of the distances of the Three of Clubs from the ends of the row. Thus, eventually the Three of Clubs is the same distance from both ends, which is to say it is the middle card, so that the illusionist is forced to remove it while it is still an interior card.

2. We construct a point X, and show that it coincides with K. Construct line ℓ through B parallel to AE. Let X be the point of intersection of ℓ with the bisector of $\angle C$. Then $\ell = BX$ is also parallel to CD, and hence

$$\begin{aligned}\angle BXC &= \angle XCD, \text{ (alternating angles)}\\ &= \angle BCX.\end{aligned}$$

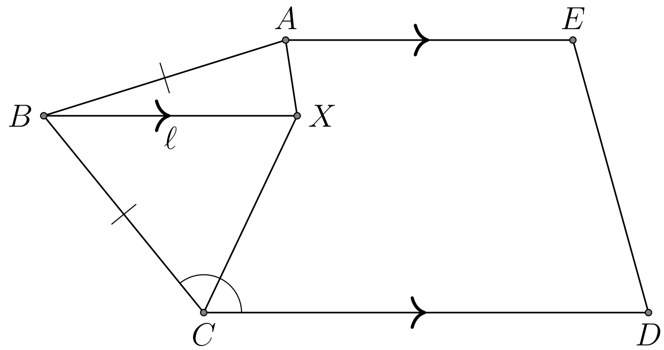
   Thus, triangle XBC is isosceles, with

$$\begin{aligned}XB &= BC\\ &= BA.\end{aligned}$$

   Hence triangle XBA is isosceles, with

$$\begin{aligned}\angle BAX &= \angle BXA\\ &= \angle XAE, \text{ (alternating angles, since } \ell = BX \parallel AE).\end{aligned}$$

   So, AX is the bisector of $\angle A$, and hence X coincides with K, the intersection of the bisectors of $\angle A$ and $\angle C$. Thus, BK is parallel to AE and the proof is complete.



3. Let f, g be the given operations, i.e. $x \overset{f}{\longmapsto} 3x+1$ and $x \overset{g}{\longmapsto} \lfloor x/2 \rfloor$. Let $P(n)$ be the statement: 'n can be written on the board'. We shall prove $P(n)$ holds for all natural numbers n by strong induction.

   The base case $P(1)$ is given, since 1 is the initial number written on the board.

   Any $n > 1$ has form $3k-1$, $3k$ or $3k+1$ for $k \geq 1$. For the inductive step, we shall deduce $P(n)$ holds, assuming $P(m)$ holds for $1 \leq m < n$. Thus, in particular, by assumption $P(k)$, $P(2k-1)$ and $P(2k)$ hold, and hence the following three map sequences:

   $$\begin{aligned} &\text{Case “}n = 3k-1\text{”: } 2k-1 \overset{f}{\longmapsto} 6k-2 \overset{g}{\longmapsto} 3k-1, \\ &\text{Case “}n = 3k\text{”: } \quad 2k \overset{f}{\longmapsto} 6k+1 \overset{g}{\longmapsto} 3k, \\ &\text{Case “}n = 3k+1\text{”: } k \overset{f}{\longmapsto} 3k+1, \end{aligned}$$

   show that $P(n)$ holds for all three forms of n, if $P(m)$ holds for $1 \leq m < n$.

   So, the induction is complete, and hence $P(n)$ holds for all natural numbers n, i.e. any natural number n can be written on the board via some sequence of f and g operations.

4. **Solution 1.** Rotate the polygon to make the sides horizontal and vertical. Let the number of the horizontal sides be k. Then, since each vertex is incident with one horizontal and one vertical edge, so that numbers of horizontal sides and vertical sides must be equal, the number of the vertical sides is also k. Note that all the vertices are of four possible types: $\ulcorner, \urcorner, \llcorner, \lrcorner$, which in order we will call, types 1 to 4.

   Since by rotating the polygon, we can alter the type of a vertex as we please, we can suppose without loss of generality that an arbitrarily chosen vertex A is type 2. Observe that the angle bisectors of vertices of types 2 and 3 are parallel, and the angle bisectors of vertices of types 1 and 4 are perpendicular to the angle bisectors of vertices of type 2. Thus the vertices that are *unfriendly* with A are the vertices of types 1 and 4.

   We claim that the total number of vertices of types 1 and 4 is even. Indeed, each vertex of type 1 or 3 is the left end of a horizontal side. Hence, the total number of vertices of types 1 and 3 is k. Let the number of type 1 vertices be n, then the number of type 3 vertices is $k-n$. Since each vertex of type 3 or 4 is the lower end of a vertical side, the total number of vertices of types 3 and 4 is k. Thus, the number of vertices of type 4 is $k-(k-n)=n$. Thus the total number of vertices of types 1 and 4, and hence the number of vertices that are *unfriendly* with A, is $2n$, which is even, as was required to be shown.

Solution 2. Rotate the polygon so that at the chosen vertex A the angle bisector ℓ is horizontal. Consider a point moving along the perimeter of the polygon with a constant speed starting and finishing at A. Then, its projection onto ℓ is also moving with a constant speed and the direction of its motion changes every time it passes through a vertex whose angle bisector is parallel to ℓ or through vertex A itself. Thus, the number of vertices whose angle bisector is parallel to ℓ is even. Since the total number of all vertices is even, the number of *unfriendly* vertices is also even.

Solution 3. Rotate the polygon to make the sides horizontal and vertical. Let the number of horizontal sides be k, then, as in Solution 1, the number of vertical sides is also k. Note that the slope of each angle bisector is either 1 or -1.

Label the vertices anticlockwise from some initial vertex with integers from 1 to $2k$ and let a_i be 1 (resp. -1) if the angle moved through at the i-th vertex is 90° (resp. 270°). As we move along the perimeter of the polygon anticlockwise, each time we pass through a vertex with $a_i = 1$ we rotate by 90° anticlockwise and each time we pass through a vertex with $a_i = -1$ we rotate by 90° clockwise. Since we rotate by 360° after one circuit of the path we find that the number m of angles of the second type is fewer by 4 than the number of angles of the first type. Hence,

$$\begin{aligned} m+m+4 &= 2k \\ m+2 &= k \\ m &= k-2, \end{aligned}$$

and so, $a_1 a_2 \cdots a_{2k} = (-1)^{k-2}$.

Observe that the slopes of the angle bisectors of two consecutive vertices are the same if and only if the angles are different. Denote the slope of the angle bisector of the i-th vertex by b_i. Without loss of generality we can assume that $a_1 = b_1$. We see that for odd i we have $a_i = b_i$ and for even i we have $a_i = -b_i$. Thus,

$$b_1 b_2 \cdots b_{2k} = (-1)^k a_1 a_2 \cdots a_{2k} = (-1)^{2k-2} = 1.$$

Hence the number of angle bisectors with slope -1 is even and, thus, the number of angle bisectors with slope 1, is also even. Therefore, whether a chosen vertex has an angle bisector of slope 1 or -1, the number of *unfriendly* vertices is even.

5. This is a "discrete optimisation" problem. We will first show that the amount of money that must be spent to rearrange the counters in reverse order is bounded below by some value B, and then we show, by explicit example, that B can be attained, where $B = \$61$. Thus we will have shown that the least amount of money that must be spent to rearrange the counters in reverse order is \$61.

   Number the squares left to right, sequentially from 1 to 100. Counters that have exactly 4 counters between them have positions k and $k+5$, respectively, for some integer k, i.e. counters whose positions are in the same residue class modulo 5, can be swapped for free. Now, colour the squares with 5 colours 0, 1, 2, 3 and 4, according to the residue class of their positions modulo 5, to achieve the pattern,

$$12340\,12340\ldots 12340.$$

To reverse the order of the counters, a counter on square k must move to square $101 - k$, or modulo 5, counters on squares coloured 0, 1, 2, 3, 4 must move to squares coloured 1, 4, 2, 3, 0, respectively.

Since we can rearrange counters that stay on squares of the same colour in any order for free, we can consider non-free swaps as those that swap a pair of counters between squares of adjacent colours. Therefore, we can reformulate the problem in the following way:

> There are five piles of counters numbered 0, 1, 2, 3, 4, placed in order around a circle. One may swap two counters between two adjacent piles for 1 dollar. We need to determine the minimum cost for swapping all the counters between the piles 0 and 1 and between the piles 2 and 4, and leaving pile 3 intact.

If every counter from pile 0 goes to pile 1, then each is involved in at least 1 non-free swap. Similarly, each counter in pile 1 is involved in at least 1 non-free swap. In moving a counter from pile 2 to pile 4, each counter is involved in at least 2 non-free swaps. Likewise, counters in pile 4 are involved in at least 2 non-free swaps. Since each non-free swap involves two counters, we have a lower bound on non-free swaps of $(20 + 20 + 40 + 40)/2 = 60$.

However, in order to achieve the objective with exactly 60 non-free swaps, each counter in pile 2 must pass through pile 3, so that at least one counter originally in pile 3 is involved in a swap. Hence the number of non-free swaps required is greater than 60.

So we have a lower bound B of \$61 for the cost of rearranging the counters in reverse order. Now we show B can be attained. Indeed, partner each counter in pile 0 with a counter in pile 1, and swap each pair; piles 0 and 1 have been swapped for a cost of \$20. Next, number the pile 2 counters: $a_1, a_2, \ldots, a_{20}$, and number the pile 3 counters: $b_1, b_2, \ldots, b_{20}$. And select one counter X in pile 3. We proceed by swapping a_1 and X, so that X is in pile 2 and a_1 in pile 3. Then swap a_1 with b_1, so that a_1 is in pile 4 and b_1 in pile 3. We continue by swapping a_2 with b_1, a_2 with b_2, and so on, until on the penultimate move we swap a_{20} with b_{20}. Finally, we swap b_{20} with X. Thus, piles 2 and 4 have been swapped, and counter X is back in pile 3, for a cost of \$41, and an overall cost of \$61. The proof is now complete.